Cantor sets and dimension
Problem set 2

1. Let $A : \{0, \dots, b-1\}^2 \to \{0,1\}$ be a $b \times b$ matrix with 0, 1 entries. Define $K_A \subset [0,1]^2$ as
$$K_A := \left\{(x,y) \in [0,1]^2 \,:\, A_{y_j,x_j} = 1, \text{ for all } j = 1,2,\dots\right\},$$
where $x = \sum x_j b^{-j}$, $y = \sum y_j b^{-j}$ are b-ary expansions. Compute $\mathrm{Hdim}(K_A)$.

2. Let $S \subset \mathbb{N}$. Define
$$A_S = \left\{ x \in [0,1] \,:\, x = \sum_{j\in S} x_j 2^{-j},\, x_j \in \{0,1\} \right\}.$$
   (a) Show that
$$\overline{\mathrm{Mdim}}(A_S) = \limsup_{n\to\infty} \frac{\#\{S \cap \{0,\dots,n\}\}}{n}, \quad \underline{\mathrm{Mdim}}(A_S) = \liminf_{n\to\infty} \frac{\#\{S \cap \{0,\dots,n\}\}}{n}.$$
   (b) Prove that $\mathrm{Pdim}(A_S) = \overline{\mathrm{Mdim}}(A_S)$. Use it to give an example of a set with
$$\overline{\mathrm{Mdim}}(A_S) = \mathrm{Pdim}(A_S) > \underline{\mathrm{Mdim}}(A_S) \geq \mathrm{Hdim}(A_S).[1]$$
   Hint: Consider a covering of A_S by closed sets and use Baire cathegory theorem to show that one of them should contain a shift of some set A_T, where $T \subset S$, and $S \setminus T$ finite.

3. Let K be the attractor for the family of uniform contractions $(T_1, T_2, \dots, T_k)$ with the contraction ratios $(r_1, r_2, \dots, r_k)$. Show that
   (a) One can find $\varepsilon_0 > 0$ and $1 > C > 0$ so that for any $x \in K$ and $\varepsilon_0 > \varepsilon > 0$, there is a sequence $j_1, \dots j_l$ such that
$$T_{j_1} \circ \dots T_{j_l} K \subset K \cap B(x,\varepsilon) \text{ and } r_{j_1} \times \cdots \times r_{j_l} > C\varepsilon.$$
   (b) If $(B(x_j,\varepsilon))_{j=1}^N$, $\varepsilon < \varepsilon_0$ is a collection of disjoint disks with $x_j \in K$ then there is an attractor $K_1 \subset K$ for some family of uniform contractions *satisfying Open Set Condition* such that
$$\mathrm{Hdim}(K_1) \geq \frac{\log N}{|\log C + \log \varepsilon|}.$$
   Hint: Use the uniform contractions constructed in the previous step for the disks $B(x_j,\varepsilon)$ to build the set K_1. Use the estimate from the previous step to estimate the similarity dimension of K_1.
   (c) Conclude that $\mathrm{Mdim}\, K$ exists and $\mathrm{Mdim}\, K = \mathrm{Hdim}\, K$.

[1] In fact, for the sets A_S we always have $\mathrm{Hdim}(A_S) = \underline{\mathrm{Mdim}}(A_S)$. It easily follows from the Billingsley's Lemma.

4. Let $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^m$. Show that

$$\mathrm{Hdim}(E) + \mathrm{Hdim}(F) \leq \mathrm{Hdim}(E \times F) \leq \mathrm{Hdim}(E) + \mathrm{Pdim}(F).$$

Hint: Use Frostman's lemma for the first inequality. For the second inequality, first prove it with $\overline{\mathrm{Mdim}}$ instead of Pdim.

5. Let $K \subset \mathbb{R}^d$ be a compact, $\alpha = d - 2$, and μ be a (positive) measure with $\mu(K) = \mu(\mathbb{R}^d) = 1$. Show that $U^\mu_\alpha(x)$ is harmonic on $\mathbb{R}^d \setminus K$.

6. Show that for any $\alpha \geq 0$ and compactly supported positive μ, the potential U^μ_α is lower-semicontinuous, i.e.

$$\liminf_{x \to x_0} U^\mu_\alpha(x) \geq U^\mu_\alpha(x_0).$$

7. (a) Show that if a sequence of measures μ_n converges to a measure μ weak*, then for any $\alpha \geq 0$, $I_\alpha(\mu) \leq \liminf_{n\to\infty} I_\alpha(\mu_n)$.
   Hint: First, you need to prove that $\mu_n \times \mu_n$ weak*-converges to $\mu \times \mu$.

   (b) Show that for any compact set K there exist a probability measure $\overline{\mu}$ supported on K with $I_\alpha(\overline{\mu}) = V_\alpha(K)$.

   (c) Show that for any probability measure ν of finite α-capacity supported on K, we have

$$< \nu, \overline{\mu} > := \int U^{\overline{\mu}}_\alpha \, d\nu(x) = \int U^\nu_\alpha \, d\overline{\mu}(x) \geq V_\alpha(K).$$

   Hint: Consider $\nu_\delta := \delta\nu + (1-\delta)\overline{\mu}$. Note that $f(\delta) := I_\alpha(\mu_\delta) \geq f(0)$.

   (d) Show that if a set E has zero α-capacity if and only if for any measure ν with finite α-energy, $\nu(E) = 0$. Use this property to show that, if $V_\alpha(K) < \infty$, the set $\left\{x : U^{\overline{\mu}}_\alpha(x) < V_\alpha(K)\right\}$ has zero α-capacity and $\overline{\mu}$ measure.

   (e) Derive that $U^{\overline{\mu}}_\alpha(x) = V_\alpha(K)$ $\overline{\mu}$-a. e.